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DIFFERENTIAL CALCULUS FOR LINEAR OPERATORS REPRESENTED BY FINITE SIGNED MEASURES AND APPLICATIONS

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Abstract. We introduce a differential calculus for linear operators represented by a family of finite signed measures. Such a calculus is based on the notions of g-derived operators and processes and g-integrating measures, g being a right-continuous nondecreasing function. Depending on the choice of g, this differential calculus works for non-smooth functions and under weak integrability conditions. For linear operators represented by stochastic processes, we provide a characterization criterion of g-differentiability in terms of characteristic functions of the random variables involved. Various illustrative examples are considered. As an application, we obtain an efficient algorithm to compute the Riemann zeta function $\zeta(z)$ with a geometric rate of convergence which improves exponentially as $\Re(z)$ increases.

1. Introduction

Derivatives of linear operators are widely used in approximation theory, particularly in dealing with strong converse inequalities (cf. Ditzian and Totik [17], Knoop and Zhou [26], Sangüesa [32], Ditzian [16], Draganov [19], Jiang and Xie [23], and the references therein), as well as in probability theory, for instance, in Poisson and binomial approximation (cf. Roos [31], Borisov and Rouzankin [9], López-Blázquez and Salamanca [28], and Barbour and Čekanavičius [8]). In many occasions, such derivatives have different forms depending on the differentiability requirements on the functions under consideration. To fix ideas, take the classical Szàsz operator L defined as

$$L\phi(t) := \sum_{k=0}^{\infty} \phi(k) \frac{e^{-t} t^k}{k!}, \qquad t \geqq 0,$$

Key words and phrases: linear operator, differential calculus, signed kernel, g-derived operator, subordinator, Riemann zeta function, efficient algorithm.
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On general fundamental solutions of some linear elliptic differential operators

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The derivation of general fundamental solutions of differential operators on tensor fields is converted, through Hörmander's method, in search of general fundamental solutions of operators on scalar fields. One resorts to the theory of distributions in order to guarantee the existence of the generalized functions required in the formulation. The procedure is applied in the determination of general fundamental solutions of some well known linear elliptic differential operators of the continuum mechanics. The study concludes that the use of general fundamental solutions can be computationally advantageous.

Key words: Boundary element method, fundamental solutions, Hörmander's method, plate bending, elasticity.

1 INTRODUCTION

The large applicability of the boundary element method (BEM) to solve engineering problems depends directly on the availability of fundamental solutions.[1,2] Although fundamental solutions are extensively described in a great number of publications, their derivation and general expressions are scarcely ever discussed. The goal of this paper is to discuss a well known procedure for the determination of general fundamental solutions of some basic linear elliptic differential operators of continuum mechanics.

At the present stage of development, the advanced application of BEM to particular problems has shown great dependence of correct interpretation and clever use of fundamental solutions in its general aspects. In this paper it is shown that fundamental solutions are formed by a combination of essential and complementary elementary functions.[3,4] Essential fundamental solutions were used extensively up to now. Nevertheless, little attention has been given to the complementary terms of fundamental solutions. This work is an attempt to cover this gap, and does not intend to be conclusive, in the sense of giving the best coefficients (or range of coefficients) of complementary functions. It should be noted that all fundamental solutions have complementary functions.[3]

The operators treated here are: Laplace, bi-harmonic (Kirchhoff's plate model), Reissner/Mindlin plate model, two- and three-dimensional elasticity. Although only applied to some operators, the procedure outlined here can be easily extended to all the family of linear elliptic differential operators with constant coefficients.

The indicial notation will be extensively used throughout this paper, with subscript greek indices in the range 1,2 and subscript roman indices in the range 1,2,3.

2 ABSTRACT THEORETICAL FOUNDATION

The establishment of the integral equations for a given physical phenomenon, starting from their mathematical description in differential form, is best performed through the application of the weighted residual method.[1] This method states that the weighting functions, according to which residual errors are minimized, are given by[1]

$$u_i^*(Q) = U_{ij}(P,Q)e_j(P)$$
$$t_i^*(Q) = T_{ij}(P,Q)e_j(P) \tag{1}$$

where $u_i^*(Q)$ and $t_i^*(Q)$ are the generalized displacements and tractions, respectively, at the field point Q

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CLASSICAL ORTHOGONAL POLYNOMIALS VIA A SECOND-ORDER LINEAR DIFFERENTIAL OPERATORS

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Let $T_c := D(x-c)((x-c)D + 2\mathbb{I})$ be a second-order linear differential operator, where c is an arbitrary complex number, $D := \frac{d}{dx}$ and $\mathbb{I}$ represents the identity on the linear space of polynomials with complex coefficients. The aim of this paper is to describe all of the T_c-classical orthogonal polynomials. Two canonical situations appear: the Laguerre $\{L_n^{(2)}\}_{n\geq 0}$ and the Jacobi $\{P_n^{(\alpha-2,2)}\}_{n\geq 0}$

Key words : Orthogonal polynomials; quasi-definite linear functionals; classical polynomials, differential operators; structure relations.

2010 Mathematics Subject Classification : 33C45, 42C05.

1. Introduction

Classical orthogonal polynomials can be characterized by means of several properties [5, 8, 24]. Among them, the so-called Hahn's characterization states that $\{P_n(x)\}_{n\geq 0}$ is a sequence of classical orthogonal polynomials if and only if the sequence of its derivatives $\{P_n'(x)\}_{n\geq 0}$ is also orthogonal. For classical orthogonal polynomials this is simple since for monic Hermite polynomials we have that $\frac{d}{dx}H_n(x) = nH_{n-1}(x)$ is again orthogonal; for monic Laguerre polynomials we have that $\frac{d}{dx}L_n^{(\alpha)}(x) = nL_{n-1}^{(\alpha+1)}(x)$ is also orthogonal; and for monic Jacobi polynomials $\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = nP_{n-1}^{(\alpha+1,\beta+1)}(x)$ is again orthogonal. Even in the case of orthogonality with respect to nonpositive

Cohomology of $\mathfrak{osp}(2|2)$ acting on the spaces of linear differential operators on the superspace $\mathbb{R}^{1|2}$

Nizar Ben Fraj   Maha Boujelben*

November 1, 2011

Abstract

We compute the first differential cohomology of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2|2)$ with coefficients in the superspace of linear differential operators acting on the space of weighted densities on the $(1,2)$-dimensional real superspace. We also compute the same, but $\mathfrak{osp}(1|2)$-relative, cohomology. We explicitly give 1-cocycles spanning these cohomology. This work is a simplest generalization of a result by Basdouri and Ben Ammar [Cohomology of $\mathfrak{osp}(1|2)$ with coefficients in $\mathfrak{D}_{\lambda,\mu}$. Lett. Math. Phys. **81**, 239–251 (2007)].

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Key words : Cohomology, Orthosymplectic superalgebra.

1 Introduction

The space of weighted densities with weight λ (or λ-densities) on $\mathbb{R}$, denoted by:

$$\mathcal{F}_\lambda = \left\{ f(dx)^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}, \quad \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes\lambda}$ for positive integer λ. Let $\mathrm{Vect}(\mathbb{R})$ be the Lie algebra of all vector fields $X_F = F\frac{d}{dx}$ on $\mathbb{R}$, where $F \in C^\infty(\mathbb{R})$. The *Lie derivative* L_D along the vector field D makes $\mathcal{F}_\lambda$ a $\mathrm{Vect}(\mathbb{R})$-module for any $\lambda \in \mathbb{R}$:

$$L_{X_F}(f(dx)^\lambda) = L^\lambda_{X_F}(f)(dx)^\lambda \quad \text{with} \quad L^\lambda_{X_F}(f) = Ff' + \lambda f F', \tag{1.1}$$

where f', F' are $\frac{df}{dx}$, $\frac{dF}{dx}$. On the space $\mathrm{D}_{\lambda,\mu}$ of differential operators $\mathcal{F}_\lambda \to \mathcal{F}_\mu$ a $\mathrm{Vect}(\mathbb{R})$-module structure is given by the formula:

$$X_F \cdot A = L^\mu_{X_F} \circ A - A \circ L^\lambda_{X_F}, \tag{1.2}$$

for any differential operator $A : f(dx)^\lambda \mapsto (Af)(dx)^\mu$.

Lecomte, in [11], found the cohomology $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathrm{D}_{\lambda,\mu})$ and $\mathrm{H}^2_{\mathrm{diff}}(\mathfrak{sl}(2), \mathrm{D}_{\lambda,\mu})$, where $\mathfrak{sl}(2)$ is realized as the Lie subalgebra of $\mathrm{Vect}(\mathbb{R})$ spanned by $\{X_1, X_x, X_{x^2}\}$ and where $\mathrm{H}^*_{\mathrm{diff}}$

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An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator

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Abstract

We discuss the problem of exhaustive enumeration of all possible factorizations for a given linear ordinary differential operator. A theoretical investigation of topological and combinatorial obstacles to uniform description of factors which include arbitrary parameters and a complete algorithm for enumeration of all (discrete and parameterized) factorizations are given.

1 Introduction

Factorization of linear ordinary differential operators (LODO)

$$L = f_0(x)D^n + f_1(x)D^{n-1} + \ldots + f_n(x), \quad D = d/dx, \tag{1}$$

$f_i(x)$ belong to some differential field $\mathbf{K}$, is a useful tool for computing a closed form solution of the corresponding LODE $Ly = 0$ as well as determining its Galois group [21, 22]. For simplicity and without loss of generality we suppose that operators are reduced (i.e. $f_0(x) \equiv 1$) unless otherwise stated explicitly. The known algorithms of factorization can provide a factorization of a LODO (over $\mathbf{K} = \overline{\mathbf{Q}}(x)$). But as the well-known example $D^2 = D \circ D = (D + 1/(x-c)) \circ (D - 1/(x-c))$ shows some LODO may have essentially different factorizations with factors depending on some arbitrary parameters. The algorithms of [8, 10, 20] are based essentially on stepwise splitting of the right factors using the old method of Beke [5] which reduces the problem of construction of a right factor $L_r = D^m + f_{r1}(x)D^{m-1} + \cdots + f_{rm}(x)$ of order m to finding "hyperexponential" solutions of the so-called *associated equation* $L_{(m)}w = 0$, i.e. solutions which have the property $f_{r1} = -Dw/w \in \mathbf{K}$. This approach fails in the case when the coefficients of L (and consequently of $L_{(m)}$) depend on parameters since the known procedures of construction of hyperexponential solutions [8, 20] of LODO obviously fail. Hence if the right factor L_r depends on parameters (they may be retrieved by the methods of [23]) we get the quotient $L_1 \circ \cdots \circ L_{r-1} = L \circ L_r^{-1}$ depending on the same parameters and shall give the parameters some definite values to proceed further obtaining only several (certainly not all in the general case) factorizations. Only in special cases the methods of [8, 10, 20] give all the possible factorizations for example if parameters do not appear or if L is completely reducible (see below sections 2, 3). An alternative approach proposed in [13] suffers from the same problem.

Fortunately according to results by Loewy [15, 16] all possible factorizations of a given (non-parametric) operator L have the same number of factors in different expansions $L = L_1 \circ \cdots \circ L_k = \overline{L}_1 \circ \cdots \circ \overline{L}_r$ into irreducible factors and the factors L_s, $\overline{L}_p$ are pairwise "similar". (Hereafter we always suppose the order of factors to be greater than 0: $\mathrm{ord}(L_i) > 0$, $\mathrm{ord}(\overline{L}_j) > 0$). Still the problem of description of all possible factorizations was unsolved in the case of factors with parameters.

In this paper we give the proper theoretical background (section 3, 4) for such exhaustive enumeration of factorizations using the Loewy-Ore [15, 16, 17, 18, 19] formal theory of LODO (section 2) and describe an algorithm for such enumeration (section 5). For simplicity we discuss here only the case of differential operators, a generalization for the case of a general Ore ring (including difference and q-difference operators, see [3, 10, 9]) is straightforward. Many of the results of this paper may be easier proved within the framework of the Picard-Vessiot theory. But we follow the formal approach of Loewy and Ore in order to facilitate the aforementioned generalizations.

2 Loewy-Ore formal theory of LODO

Here we sketch the basics of this theory [15, 16, 17, 18, 19] necessary for sections 3, 4, 5. For any two LODO L and M from the existence of the Euclid algorithm one can determine their right greatest common divisor $rGCD(L, M) = G$, i.e. $L = L_1 \circ G$, $M = M_1 \circ G$ (the order of G is maximal) and their right least common multiple $rLCM(L, M) = K$, i.e. $K = \overline{M} \circ L = \overline{L} \circ M$ (the order of K is minimal) as well as

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[total of 83 entries: 1-25 | 26-50 | 51-75 | 76-83] [showing 25 entries per page: fewer | more | all] [1] arXiv:2208.08974 [pdf, ps, other] [2] arXiv:2208.08955 [pdf, ps, other] [3] arXiv:2208.08928 [pdf, ps, other] [4] arXiv:2208.08822 [pdf, ps, other] [5] arXiv:2208.08720 [pdf, ps, other] [6] arXiv:2208.08657 [pdf, ps, other] [7] arXiv:2208.08653 [pdf, other] [8] arXiv:2208.08536 [pdf, other] [9] arXiv:2208.08526 [pdf, other] [10] arXiv:2208.08510 [pdf, ps, other] [11] arXiv:2208.08702 (cross-list from gr-qc) [pdf, ps, other] [12] arXiv:2208.08676 (cross-list from gr-qc) [pdf, ps, other] [13] arXiv:2208.07894 (cross-list from math-ph) [pdf, ps, other] [14] arXiv:2208.08378 [pdf, ps, other] [15] arXiv:2208.08360 [pdf, ps, other] [16] arXiv:2208.08334 [pdf, ps, other] [17] arXiv:2208.08328 [pdf, ps, other] [18] arXiv:2208.08317 [pdf, ps, other] [19] arXiv:2208.08312 [pdf, ps, other] [20] arXiv:2208.08311 [pdf, other] [21] arXiv:2208.08177 [pdf, ps, other] [22] arXiv:2208.08164 [pdf, ps, other] [23] arXiv:2208.08152 [pdf, ps, other] [24] arXiv:2208.08103 [pdf, ps, other] [25] arXiv:2208.08066 [pdf, ps, other] [total of 83 entries: 1-25 | 26-50 | 51-75 | 76-83] [showing 25 entries per page: fewer | more | all] Disable MathJax (What is MathJax?) Links to: arXiv, form interface, find, math, new, 2208, contact, help (Access key information) Differential equations that are linear with respect to the unknown function and its derivatives This article is about linear differential equations with one independent variable. For similar equations with two or more independent variables, see Partial differential equation § Linear equations of second order. Differential equationsNavier–Stokes differential equations used to simulate airflow around an obstruction Scope Fields Natural sciencesEngineering Astronomy Physics Chemistry Biology Geology Applied mathematics Continuum mechanics Chaos theory Dynamical systems Social sciences Economics Population dynamics Classification Types Ordinary Partial Differential-algebraic Integro-differential Fractional Linear Non-linear By variable type Dependent and independent variables Autonomous Coupled / Decoupled Exact Homogeneous / Nonhomogeneous Features Order Operator Notation Relation to processes Difference (discrete analogue) Stochastic Stochastic partial Delay Solution Existence and uniqueness Picard–Lindelöf theorem Peano existence theorem Carathéodory's existence theorem Cauchy–Kowalevski theorem General topics Wronskian Phase portrait Phase space Lyapunov / Asymptotic / Exponential stability Rate of convergence Series / Integral solutions Numerical integration Dirac delta function Solution methods Inspection Method of characteristics Euler Exponential response formula Finite difference (Crank–Nicolson) Finite element Infinite element Finite volume Galerkin Petrov–Galerkin Integrating factor Integral transforms Perturbation theory Runge–Kutta Separation of variables Undetermined coefficients Variation of parameters People List Isaac Newton Joseph Fourier Gottfried Leibniz Leonhard Euler Émile Picard Józef Maria Hoene-Wroński Ernst Lindelöf Rudolf Lipschitz Augustin-Louis Cauchy John Crank Phyllis Nicolson Carl David Tolmé Runge Martin Kutta vte In mathematics, a linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives, that is an equation of the form a 0 (x) y + a 1 (x) y ′ + a 2 (x) y ″ ⋯ + a n (x) y (n) = b (x) {\displaystyle a_{0}(x)y+a_{1}(x)y'+a_{2}(x)y''\cdots +a_{n}(x)y^{(n)}=b(x)} where a0(x), ..., an(x) and b(x) are arbitrary differentiable functions that do not need to be linear, and y′, ..., y(n) are the successive derivatives of an unknown function y of the variable x. Such an equation is an ordinary differential equation (ODE). A linear differential equation may also be a linear partial differential equation (PDE), if the unknown function depends on several variables, and the derivatives that appear in the equation are partial derivatives. A linear differential equation or a system of linear equations such that the associated homogeneous equations have constant coefficients may be solved by quadrature, which means that the solutions may be expressed in terms of integrals. This is also true for a linear equation of order one, with non-constant coefficients. An equation of order two or higher with non-constant coefficients cannot, in general, be solved by quadrature. For order two, Kovacic's algorithm allows deciding whether there are solutions in terms of integrals, and computing them if any. The solutions of homogeneous linear differential equations with polynomial coefficients are called holonomic functions. This class of functions is stable under sums, products, differentiation, integration, and contains many usual functions and special functions such as exponential function, logarithm, sine, cosine, inverse trigonometric functions, error function, Bessel functions and hypergeometric functions. Their representation by the defining differential equation and initial conditions allows making algorithmic (on these functions) most operations of calculus, such as computation of antiderivatives, limits, asymptotic expansion, and numerical evaluation to any precision, with a certified error bound. Basic terminology The highest order of derivation that appears in a (linear) differential equation is the order of the equation. The term b(x), which does not depend on the unknown function and its derivatives, is sometimes called the constant term of the equation (by analogy with algebraic equations), even when this term is a non-constant function. If the constant term is the zero function, then the differential equation is said to be homogeneous, as it is a homogeneous polynomial in the unknown function and its derivatives. The equation obtained by replacing, in a linear differential equation, the constant term by the zero function is the associated homogeneous equation. A differential equation has constant coefficients if only constant functions appear as coefficients in the associated homogeneous equation. A solution of a differential equation is a function that satisfies the equation. The solutions of a homogeneous linear differential equation form a vector space. In the ordinary case, this vector space has a finite dimension, equal to the order of the equation. All solutions of a linear differential equation are found by adding to a particular solution any solution of the associated homogeneous equation. Linear differential operator Main article: Differential operator A basic differential operator of order i is a mapping that maps any differentiable function to its ith derivative, or, in the case of several variables, to one of its partial derivatives of order i. It is commonly denoted d i d x i {\displaystyle {\frac {d^{i}}{dx^{i}}}} in the case of univariate functions, and ∂ i 1 + ⋯ + i n ∂ x 1 i 1 ⋯ ∂ x n i n {\displaystyle {\frac {\partial ^{i_{1}+\cdots +i_{n}}}{\partial x_{1}^{i_{1}}\cdots \partial x_{n}^{i_{n}}}}} in the case of functions of n variables. The basic differential operators include the derivative of order 0, which is the identity mapping. A linear differential operator (abbreviated, in this article, as linear operator or, simply, operator) is a linear combination of basic differential operators, with differentiable functions as coefficients. In the univariate case, a linear operator has thus the form[1] a 0 (x) + a 1 (x) d d x + ⋯ + a n (x) d n d x n , {\displaystyle a_{0}(x)+a_{1}(x){\frac {d}{dx}}+\cdots +a_{n}(x){\frac {d^{n}}{dx^{n}}},} where a0(x), ..., an(x) are differentiable functions, and the nonnegative integer n is the order of the operator (if an(x) is not the zero function). Let L be a linear differential operator. The application of L to a function f is usually denoted Lf or Lf(X), if one needs to specify the variable (this must not be confused with a multiplication). A linear differential operator is a linear operator, since it maps sums to sums and the product by a scalar to the product by the same scalar. As the sum of two linear operators is a linear operator, as well as the product (on the left) of a linear operator by a differentiable function, the linear differential operators form a vector space over the real numbers or the complex numbers (depending on the nature of the functions that are considered). They form also a free module over the ring of differentiable functions. The language of operators allows a compact writing for differentiable equations: if L = a 0 (x) + a 1 (x) d d x + ⋯ + a n (x) d n d x n , {\displaystyle L=a_{0}(x)+a_{1}(x){\frac {d}{dx}}+\cdots +a_{n}(x){\frac {d^{n}}{dx^{n}}},} is a linear differential operator, then the equation a 0 (x) y + a 1 (x) y ′ + a 2 (x) y ″ + ⋯ + a n (x) y (n) = b (x) {\displaystyle a_{0}(x)y+a_{1}(x)y'+a_{2}(x)y''+\cdots +a_{n}(x)y^{(n)}=b(x)} may be rewritten L y = b (x) . {\displaystyle Ly=b(x).} There may be several variants to this notation; in particular the variable of differentiation may appear explicitly or not in y and the right-hand and of the equation, such as Ly(x) = b(x) or Ly = b. The kernel of a linear differential operator is its kernel as a linear mapping, that is the vector space of the solutions of the (homogeneous) differential equation Ly = 0. In the case of an ordinary differential operator of order n, Carathéodory's existence theorem implies that, under very mild conditions, the kernel of L is a vector space of dimension n, and that the solutions of the equation Ly(x) = b(x) have the form S 0 (x) + c 1 S 1 (x) + ⋯ + c n S n (x) , {\displaystyle S_{0}(x)+c_{1}S_{1}(x)+\cdots +c_{n}S_{n}(x),} where c1, ..., cn are arbitrary numbers. Typically, the hypotheses of Carathéodory's theorem are satisfied in an interval I, if the functions b, a0, ..., an are continuous in I, and there is a positive real number k such that |an(x)| > k for every x in I. Homogeneous equation with constant coefficients A homogeneous linear differential equation has constant coefficients if it has the form a 0 y + a 1 y ′ + a 2 y ″ + ⋯ + a n y (n) = 0 {\displaystyle a_{0}y+a_{1}y'+a_{2}y''+\cdots +a_{n}y^{(n)}=0} where a1, ..., an are (real or complex) numbers. In other words, it has constant coefficients if it is defined by a linear operator with constant coefficients. The study of these differential equations with constant coefficients dates back to Leonhard Euler, who introduced the exponential function ex, which is the unique solution of the equation f′ = f such that f(0) = 1. It follows that the nth derivative of ecx is cnecx, and this allows solving homogeneous linear differential equations rather easily. Let a 0 y + a 1 y ′ + a 2 y ″ + ⋯ + a n y (n) = 0 {\displaystyle a_{0}y+a_{1}y'+a_{2}y''+\cdots +a_{n}y^{(n)}=0} be a homogeneous linear differential equation with constant coefficients (that is a0, ..., an are real or complex numbers). Searching solutions of this equation that have the form eαx is equivalent to searching the constants α such that a 0 e α x + a 1 α e α x + a 2 α 2 e α x + ⋯ + a n α n e α x = 0. {\displaystyle a_{0}e^{\alpha x}+a_{1}\alpha e^{\alpha x}+a_{2}\alpha ^{2}e^{\alpha x}+\cdots +a_{n}\alpha ^{n}e^{\alpha x}=0.} Factoring out eαx (which is never zero), shows that α must be a root of the characteristic polynomial a 0 + a 1 t + a 2 t 2 + ⋯ + a n t n {\displaystyle a_{0}+a_{1}t+a_{2}t^{2}+\cdots +a_{n}t^{n}} of the differential equation, which is the left-hand side of the characteristic equation a 0 + a 1 t + a 2 t 2 + ⋯ + a n t n = 0. {\displaystyle a_{0}+a_{1}t+a_{2}t^{2}+\cdots +a_{n}t^{n}=0.} When these roots are all distinct, one has n distinct solutions that are not necessarily real, even if the coefficients of the equation are real. These solutions can be shown to be linearly independent, by considering the Vandermonde determinant of the values of these solutions at x = 0, ..., n – 1. Together they form a basis of the vector space of solutions of the differential equation (that is, the kernel of the differential operator). Example y ⁗ − 2 y ‴ + 2 y ″ − 2 y ′ + y = 0 {\displaystyle y''''-2y'''+2y''-2y'+y=0} has the characteristic equation z 4 − 2 z 3 + 2 z 2 − 2 z + 1 = 0. {\displaystyle z^{4}-2z^{3}+2z^{2}-2z+1=0.} This has zeros, i, −i, and 1 (multiplicity 2). The solution basis is thus e i x , e − i x , e x , x e x . {\displaystyle e^{ix},\;e^{-ix},\;e^{x},\;xe^{x}.} A real basis of solution is thus cos x , sin x , e x , x e x . {\displaystyle \cos x,\;\sin x,\;e^{x},\;xe^{x}.} In the case where the characteristic polynomial has only simple roots, the preceding provides a complete basis of the solutions vector space. In the case of multiple roots, more linearly independent solutions are needed for having a basis. These have the form x k e α x , {\displaystyle x^{k}e^{\alpha x},} where k is a nonnegative integer, α is a root of the characteristic polynomial of multiplicity m, and k < m. For proving that these functions are solutions, one may remark that if α is a root of the characteristic polynomial of multiplicity m, the characteristic polynomial may be factored as P(t)(t − α)m. Thus, applying the differential operator of the equation is equivalent with applying first m times the operator d d x − α {\textstyle {\frac {d}{dx}}-\alpha } , and then the operator that has P as characteristic polynomial. By the exponential shift theorem, (d d x − α) (x k e α x) = k x k − 1 e α x , {\displaystyle \left({\frac {d}{dx}}-\alpha \right)\left(x^{k}e^{\alpha x}\right)=kx^{k-1}e^{\alpha x},} and thus one gets zero after k + 1 application of d d x − α {\textstyle {\frac {d}{dx}}-\alpha } . As, by the fundamental theorem of algebra, the sum of the multiplicities of the roots of a polynomial equals the degree of the polynomial, the number of above solutions equals the order of the differential equation, and these solutions form a base of the vector space of the solutions. In the common case where the coefficients of the equation are real, it is generally more convenient to have a basis of the solutions consisting of real-valued functions. Such a basis may be obtained from the preceding basis by remarking that, if a + ib is a root of the characteristic polynomial, then a – ib is also a root, of the same multiplicity. Thus a real basis is obtained by using Euler's formula, and replacing x k e (a + i b) x {\displaystyle x^{k}e^{(a+ib)x}} and x k e (a − i b) x {\displaystyle x^{k}e^{(a-ib)x}} by x k e a x cos (b x) {\displaystyle x^{k}e^{ax}\cos(bx)} and x k e a x sin (b x) {\displaystyle x^{k}e^{ax}\sin(bx)} . Second-order case A homogeneous linear differential equation of the second order may be written y ″ + a y ′ + b y = 0 , {\displaystyle y''+ay'+by=0,} and its characteristic polynomial is r 2 + a r + b . {\displaystyle r^{2}+ar+b.} If a and b are real, there are three cases for the solutions, depending on the discriminant D = a2 − 4b. In all three cases, the general solution depends on two arbitrary constants c1 and c2. If D > 0, the characteristic polynomial has two distinct real roots α, and β. In this case, the general solution is c 1 e α x + c 2 e β x . {\displaystyle c_{1}e^{\alpha x}+c_{2}e^{\beta x}.} If D = 0, the characteristic polynomial has a double root −a/2, and the general solution is (c 1 + c 2 x) e − a x / 2 . {\displaystyle (c_{1}+c_{2}x)e^{-ax/2}.} If D < 0, the characteristic polynomial has two complex conjugate roots α ± βi, and the general solution is c 1 e (α + β i) x + c 2 e (α − β i) x , {\displaystyle c_{1}e^{(\alpha +\beta i)x}+c_{2}e^{(\alpha -\beta i)x},} which may be rewritten in real terms, using Euler's formula as e α x (c 1 cos (β x) + c 2 sin (β x)) . {\displaystyle e^{\alpha x}(c_{1}\cos(\beta x)+c_{2}\sin(\beta x)).} Finding the solution y(x) satisfying y(0) = d1 and y′(0) = d2, one equates the values of the above general solution at 0 and its derivative there to d1 and d2, respectively. This results in a linear system of two linear equations in the two unknowns c1 and c2. Solving this system gives the solution for a so-called Cauchy problem, in which the values at 0 for the solution of the DEQ and its derivative are specified. Non-homogeneous equation with constant coefficients A non-homogeneous equation of order n with constant coefficients may be written y (n) (x) + a 1 y (n − 1) (x) + ⋯ + a n − 1 y ′ (x) + a n y (x) = f (x) , {\displaystyle y^{(n)}(x)+a_{1}y^{(n-1)}(x)+\cdots +a_{n-1}y'(x)+a_{n}y(x)=f(x),} where a1, ..., an are real or complex numbers, f is a given function of x, and y is the unknown function (for sake of simplicity, "(x)" will be omitted in the following). There are several methods for solving such an equation. The best method depends on the nature of the function f that makes the equation non-homogeneous. If f is a linear combination of exponential and sinusoidal functions, then the exponential response formula may be used. If, more generally, f is a linear combination of functions of the form xneax, xn cos(ax), and xn sin(ax), where n is a nonnegative integer, and a a constant (which need not be the same in each term), then the method of undetermined coefficients may be used. Still more general, the annihilator method applies when f satisfies a homogeneous linear differential equation, typically, a holonomic function. The most general method is the variation of constants, which is presented here. The general solution of the associated homogeneous equation y (n) + a 1 y (n − 1) + ⋯ + a n − 1 y ′ + a n y = 0 {\displaystyle y^{(n)}+a_{1}y^{(n-1)}+\cdots +a_{n-1}y'+a_{n}y=0} is y = u 1 y 1 + ⋯ + u n y n , {\displaystyle y=u_{1}y_{1}+\cdots +u_{n}y_{n},} where (y1, ..., yn) is a basis of the vector space of the solutions and u1, ..., un are arbitrary constants. The method of variation of constants takes its name from the following idea. Instead of considering u1, ..., un as constants, they can be considered as unknown functions that have to be determined for making y a solution of the non-homogeneous equation. For this purpose, one adds the constraints 0 = u 1 ′ y 1 + u 2 ′ y 2 + ⋯ + u n ′ y n 0 = u 1 ′ y 1 ′ + u 2 ′ y 2 ′ + ⋯ + u n ′ y n ′ ⋮ 0 = u 1 ′ y 1 (n − 2) + u 2 ′ y 2 (n − 2) + ⋯ + u n ′ y n (n − 2) , {\displaystyle {\begin{aligned}0&=u'_{1}y_{1}+u'_{2}y_{2}+\cdots +u'_{n}y_{n}\\0&=u'_{1}y'_{1}+u'_{2}y'_{2}+\cdots +u'_{n}y'_{n}\\&\;\;\vdots \\0&=u'_{1}y_{1}^{(n-2)}+u'_{2}y_{2}^{(n-2)}+\cdots +u'_{n}y_{n}^{(n-2)},\end{aligned}}} which imply (by product rule and induction) y (i) = u 1 y 1 (i) + ⋯ + u n y n (i) {\displaystyle y^{(i)}=u_{1}y_{1}^{(i)}+\cdots +u_{n}y_{n}^{(i)}} for i = 1, ..., n – 1, and y (n) = u 1 y 1 (n) + ⋯ + u n y n (n) + u 1 ′ y 1 (n − 1) + u 2 ′ y 2 (n − 1) + ⋯ + u n ′ y n (n − 1) . {\displaystyle y^{(n)}=u_{1}y_{1}^{(n)}+\cdots +u_{n}y_{n}^{(n)}+u'_{1}y_{1}^{(n-1)}+u'_{2}y_{2}^{(n-1)}+\cdots +u'_{n}y_{n}^{(n-1)}.} Replacing in the original equation y and its derivatives by these expressions, and using the fact that y1, ..., yn are solutions of the original homogeneous equation, one gets f = u 1 ′ y 1 (n − 1) + ⋯ + u n ′ y n (n − 1) . {\displaystyle f=u'_{1}y_{1}^{(n-1)}+\cdots +u'_{n}y_{n}^{(n-1)}.} This equation and the above ones with 0 as left-hand side form a system of n linear equations in u′1, ..., u′n whose coefficients are known functions (f, the yi, and their derivatives). This system can be solved by any method of linear algebra. The computation of antiderivatives gives u1, ..., un, and then y = u1y1 + ⋯ + unyn. As antiderivatives are defined up to the addition of a constant, one finds again that the general solution of the non-homogeneous equation is the sum of an arbitrary solution and the general solution of the associated homogeneous equation. First-order equation with variable coefficients The general form of a linear ordinary differential equation of order 1, after dividing out the coefficient of y′(x), is: y ′ (x) = f (x) y (x) + g (x) . {\displaystyle y'(x)=f(x)y(x)+g(x).} If the equation is homogeneous, i.e. g(x) = 0, one may rewrite and integrate: y ′ y = f , log y = k + F , {\displaystyle {\frac {y'}{y}}=f,\qquad \log y=k+F,} where k is an arbitrary constant of integration and F = ∫ f d x {\displaystyle F=\textstyle \int f\,dx} is any antiderivative of f. Thus, the general solution of the homogeneous equation is y = c e F , {\displaystyle y=ce^{F},} where c = ek is an arbitrary constant. For the general non-homogeneous equation, one may multiply it by the reciprocal e−F of a solution of the homogeneous equation.[2] This gives y ′ e − F − y f e − F = g e − F . {\displaystyle y'e^{-F}-yfe^{-F}=ge^{-F}.} As − f e − F = d d x (e − F) , {\displaystyle -fe^{-F}={\tfrac {d}{dx}}\left(e^{-F}\right),} the product rule allows rewriting the equation as d d x (y e − F) = g e − F . {\displaystyle {\frac {d}{dx}}\left(ye^{-F}\right)=ge^{-F}.} Thus, the general solution is y = c e F + e F ∫ g e − F d x , {\displaystyle y=ce^{F}+e^{F}\int ge^{-F}dx,} where c is a constant of integration, and F is any antiderivative of f (changing of antiderivative amounts to change the constant of integration). Example Solving the equation y ′ (x) + y (x) x = 3 x . {\displaystyle y'(x)+{\frac {y(x)}{x}}=3x.} The associated homogeneous equation y ′ (x) + y (x) x = 0 {\displaystyle y'(x)+{\frac {y(x)}{x}}=0} gives y ′ y = − 1 x , {\displaystyle {\frac {y'}{y}}=-{\frac {1}{x}},} that is y = c x . {\displaystyle y={\frac {c}{x}}.} Dividing the original equation by one of these solutions gives x y ′ + y = 3 x 2 . {\displaystyle xy'+y=3x^{2}.} That is (x y) ′ = 3 x 2 , {\displaystyle (xy)'=3x^{2},} x y = x 3 + c , {\displaystyle xy=x^{3}+c,} and y (x) = x 2 + c / x . {\displaystyle y(x)=x^{2}+c/x.} For the initial condition y (1) = α , {\displaystyle y(1)=\alpha ,} one gets the particular solution y (x) = x 2 + α − 1 x . {\displaystyle y(x)=x^{2}+{\frac {\alpha -1}{x}}.} System of linear differential equations Main article: Matrix differential equation A system of linear differential equations consists of several linear differential equations that involve several unknown functions. In general one restricts the study to systems such that the number of unknown functions equals the number of equations. An arbitrary linear ordinary differential equation and a system of such equations can be converted into a first order system of linear differential equations by adding variables for all but the highest order derivatives. That is, if y ′ , y ″ , … , y (k) {\displaystyle y',y'',\ldots ,y^{(k)}} appear in an equation, one may replace them by new unknown functions y 1 , … , y k {\displaystyle y_{1},\ldots ,y_{k}} that must satisfy the equations y ′ = y 1 {\displaystyle y'=y_{1}} and y i ′ = y i + 1 , {\displaystyle y_{i}'=y_{i+1},} for i = 1, ..., k – 1. A linear system of the first order, which has n unknown functions and n differential equations may normally be solved for the derivatives of the unknown functions. If it is not the case this is a differential-algebraic system, and this is a different theory. Therefore, the systems that are considered here have the form y 1 ′ (x) = b 1 (x) + a 1 , 1 (x) y 1 + ⋯ + a 1 , n (x) y n ⋮ y n ′ (x) = b n (x) + a n , 1 (x) y 1 + ⋯ + a n , n (x) y n , {\displaystyle {\begin{aligned}y_{1}'(x)&=b_{1}(x)+a_{1,1}(x)y_{1}+\cdots +a_{1,n}(x)y_{n}\\\vdots &\\y_{n}'(x)&=b_{n}(x)+a_{n,1}(x)y_{1}+\cdots +a_{n,n}(x)y_{n},\end{aligned}}} where b n {\displaystyle b_{n}} and the a i , j {\displaystyle a_{i,j}} are functions of x. In matrix notation, this system may be written (omitting "(x)") y ′ = A y + b . {\displaystyle \mathbf {y} '=A\mathbf {y} +\mathbf {b} .} The solving method is similar to that of a single first order linear differential equations, but with complications stemming from noncommutativity of matrix multiplication. Let u ′ = A u . {\displaystyle \mathbf {u} '=A\mathbf {u} .} be the homogeneous equation associated to the above matrix equation. Its solutions form a vector space of dimension n, and are therefore the columns of a square matrix of functions U (x) {\displaystyle U(x)} , whose determinant is not the zero function. If n = 1, or A is a matrix of constants, or, more generally, if A commutes with its antiderivative B = ∫ A d x {\displaystyle \textstyle B=\int Adx} , then one may choose U equal the exponential of B. In fact, in these cases, one has d d x exp ⁡ (B) = A exp ⁡ (B) . {\displaystyle {\frac {d}{dx}}\exp(B)=A\exp(B).} In the general case there is no closed-form solution for the homogeneous equation, and one has to use either a numerical method, or an approximation method such as Magnus expansion. Knowing the matrix U, the general solution of the non-homogeneous equation is y (x) = U (x) y 0 + U (x) ∫ U − 1 (x) b (x) d x , {\displaystyle \mathbf {y} (x)=U(x)\mathbf {y_{0}} +U(x)\int U^{-1}(x)\mathbf {b} (x)\,dx,} where the column matrix y 0 {\displaystyle \mathbf {y_{0}} } is an arbitrary constant of integration. If initial conditions are given as y (x 0) = y 0 , {\displaystyle \mathbf {y} (x_{0})=\mathbf {y} _{0},} the solution that satisfies these initial conditions is y (x) = U (x) U − 1 (x 0) y 0 + U (x) ∫ x 0 x U − 1 (t) b (t) d t . {\displaystyle \mathbf {y} (x)=U(x)U^{-1}(x_{0})\mathbf {y_{0}} +U(x)\int _{x_{0}}^{x}U^{-1}(t)\mathbf {b} (t)\,dt.} Higher order with variable coefficients A linear ordinary equation of order one with variable coefficients may be solved by quadrature, which means that the solutions may be expressed in terms of integrals. This is not the case for order at least two. This is the main result of Picard–Vessiot theory which was initiated by Émile Picard and Ernest Vessiot, and whose recent developments are called differential Galois theory. The impossibility of solving by quadrature can be compared with the Abel–Ruffini theorem, which states that an algebraic equation of degree at least five cannot, in general, be solved by radicals. This analogy extends to the proof methods and motivates the denomination of differential Galois theory. Similarly to the algebraic case, the theory allows deciding which equations may be solved by quadrature, and if possible solving them. However, for both theories, the necessary computations are extremely difficult, even with the most powerful computers. Nevertheless, the case of order two with rational coefficients has been completely solved by Kovacic's algorithm. Cauchy–Euler equation Cauchy–Euler equations are examples of equations of any order, with variable coefficients, that can be solved explicitly. These are the equations of the form x n y (n) (x) + a n − 1 x n − 1 y (n − 1) (x) + ⋯ + a 0 y (x) = 0 , {\displaystyle x^{n}y^{(n)}(x)+a_{n-1}x^{n-1}y^{(n-1)}(x)+\cdots +a_{0}y(x)=0,} where a 0 , … , a n − 1 {\displaystyle a_{0},\ldots ,a_{n-1}} are constant coefficients. Holonomic functions Main article: holonomic function A holonomic function, also called a D-finite function, is a function that is a solution of a homogeneous linear differential equation with polynomial coefficients. Most functions that are commonly considered in mathematics are holonomic or quotients of holonomic functions. In fact, holonomic functions include polynomials, algebraic functions, logarithm, exponential function, sine, cosine, hyperbolic sine, hyperbolic cosine, inverse trigonometric and inverse hyperbolic functions, and many special functions such as Bessel functions and hypergeometric functions. Holonomic functions have several closure properties; in particular, sums, products, derivative and integrals of holonomic functions are holonomic. Moreover, these closure properties are effective, in the sense that there are algorithms for computing the differential equation of the result of any of these operations, knowing the differential equations of the input.[3] Usefulness of the concept of holonomic functions results of Zeilberger's theorem, which follows.[3] A holonomic sequence is a sequence of numbers that may be generated by a recurrence relation with polynomial coefficients. The coefficients of the Taylor series at a point of a holonomic function form a holonomic sequence. Conversely, if the sequence of the coefficients of a power series is holonomic, then the series defines a holonomic function (even if the radius of convergence is zero). There are efficient algorithms for both conversions, that is for computing the recurrence relation from the differential equation, and vice versa. [3] It follows that, if one represents (in a computer) holonomic functions by their defining differential equations and initial conditions, most calculus operations can be done automatically on these functions, such as derivative, indefinite and definite integral, fast computation of Taylor series (thanks of the recurrence relation on its coefficients), evaluation to a high precision with certified bound of the approximation error, limits, localization of singularities, asymptotic behavior at infinity and near singularities, proof of identities, etc.[4] See also Continuous-repayment mortgage Fourier transform Laplace transform Linear difference equation Variation of parameters References ^ Gershenfeld 1999, p.9 ^ Motivation: In analogy to completing the square, we write the equation as y′ − fy = g, and try to modify the left side so it becomes a derivative. Specifically, we seek an "integrating factor" h = h(x) such that multiplying by it makes the left side equal to the derivative of hy, namely hy′ − hfy = (hy)′. This means h′ = −f, so that h = e−∫ f dx = e−F, as in the text. ^ a b c Zeilberger, Doron. A holonomic systems approach to special functions identities. Journal of computational and applied mathematics. 32.3 (1990): 321-368 ^ Benoit, A., Chyzak, F., Darrasse, A., Gerhold, S., Mezzarobba, M., & Salvy, B. (2010, September). The dynamic dictionary of mathematical functions (DDMF). In International Congress on Mathematical Software (pp. 35-41). Springer, Berlin, Heidelberg. Birkhoff, Garrett & Rota, Gian-Carlo (1978), Ordinary Differential Equations, New York: John Wiley and Sons, Inc., ISBN 0-471-07411-X Gershenfeld, Neil (1999), The Nature of Mathematical Modeling, Cambridge, UK.: Cambridge University Press, ISBN 978-0-521-57095-4 Robinson, James C. 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